Local Bianchi Identities in the Relativistic Non-Autonomous Lagrange Geometry

Mircea Neagu

Abstract

The aim of this paper is to describe the local Bianchi identities for an h-normal Γ-linear connection of Cartan type $\nabla\Gamma$ on the first-order jet space $J^1(\mathbb{R}, M)$. In this direction, we present the local expressions of the adapted components of the torsion and curvature d-tensors produced by $\nabla\Gamma$ and we give the general local expressions of Bianchi identities which connect these d-torsions and d-curvatures.

Mathematics Subject Classification (2000): 53C60, 53C43, 53C07.

Key words and phrases: the 1-jet space $J^1(\mathbb{R}, M)$, nonlinear connections, d-linear connections of Cartan type, d-torsions and d-curvatures, Bianchi identities.

1 Introduction

It is well known that the 1-jet spaces are basic objects in the study of classical and quantum field theories. For such a reason, a lot of authors (Asanov [2], Krupková [7], Saunders [17], Vondra [18] and many others) studied the differential geometry of 1-jet spaces. Going on the geometrical studies initiated by Asanov [2] and using as a pattern the Lagrangian geometrical ideas developed by Miron, Anastasiei and Bucătaru in the monographs [9] and [4], the author of this paper has recently developed the *Riemann-Lagrange geometry of the* 1*-jet spaces* [13]. This is a general geometrical framework for the study of ***relativistic non-autonomous (rheonomic*** or ***time dependent) Lagrangians*** [14] or *relativistic multi-time dependent Lagrangians* [13].

We underline that a ***classical non-autonomous (rheonomic*** or ***time dependent) Lagrangian geometry***, that is a geometrization of Lagrangians depending on an ***absolute time***, was sketched at level of ideas by Miron and Anastasiei in the last chapter of the book [9]. That classical non-autonomous Lagrangian geometry was developed further by Anastasiei and Kawaguchi [1] or Frigioiu [6].

In what follows, we try to expose the main geometrical and physical aspects which differentiate the both geometrical theories: the ***jet relativistic non-autonomous Lagrangian geometry*** [14] and the ***classical non-autonomous Lagrangian geometry*** [9].

In this way, we point out that the ***relativistic non-autonomous Lagrangian geometry*** [14] has as natural house the 1-jet space $J^1(\mathbb{R}, M)$, where $\mathbb{R}$ is the manifold of real numbers having the coordinate t. This coordinate represents for us the usual *relativistic time*. We recall that the 1-jet space $J^1(\mathbb{R}, M)$ is regarded as a vector bundle over the product manifold $\mathbb{R} \times M$, having the fibre type $\mathbb{R}^n$, where n is the dimension of the *spatial* manifold M. In mechanical terms, if the manifold M has the spatial local coordinates $(x^i)_{i=\overline{1,n}}$, then the 1-jet vector bundle $J^1(\mathbb{R}, M) \to \mathbb{R} \times M$ can be regarded as a *bundle of configurations* having the local coordinates (t, x^i, y_1^i); these transform by the rules [11]

$$\begin{cases} \widetilde{t} = \widetilde{t}(t) \\ \widetilde{x}^i = \widetilde{x}^i(x^j) \\ \widetilde{y}_1^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j} \dfrac{dt}{d\widetilde{t}} \cdot y_1^j. \end{cases} \tag{1.1}$$

Remark 1.1 *The form of the jet transformation group (1.1) stands out by the **relativistic** character of the **time** t.*

Comparatively, the ***classical non-autonomous Lagrangian geometry*** from [9] has as *bundle of configurations* the vector bundle $\mathbb{R} \times TM \to M$, whose local coordinates (t, x^i, y^i) transform by the rules

$$\begin{cases} \widetilde{t} = t \\ \widetilde{x}^i = \widetilde{x}^i(x^j) \\ \widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j} \cdot y^j, \end{cases} \tag{1.2}$$

where TM is the tangent bundle of the spatial manifold M.

Remark 1.2 *The form of the transformation group (1.2) stands out by the **absolute** character of the **time** t.*

It is obvious that the jet transformation group (1.1) from the *relativistic non-autonomous Lagrangian geometry* is more general and more natural than the transformation group (1.2) used in the *classical non-autonomous Lagrangian geometry*. The naturalness of the transformation group (1.1) comes from the fact that the relativity of time is well-known. In contrast, the transformation group (1.2) ignores the temporal reparametrizations, emphasizing in this way the absolute character of the usual time coordinate t.

From a geometrical point of view, we point out that the whole *classical non-autonomous Lagrangian geometry* initiated by Miron and Anastasiei [9] relies on the study of the *absolute energy action functional*

$$\mathbb{E}_1(c) = \int_a^b L(t, x^i, y^i)dt,$$

where $L : \mathbb{R} \times TM \to \mathbb{R}$ is a Lagrangian function and $y^i = dx^i/dt$, whose Euler-Lagrange equations produce a semispray $G^i(t, x^k, y^k)$ and a corresponding nonlinear connection

$$N_j^i = \frac{\partial G^i}{\partial y^j}.$$

Therefore, the authors construct the adapted bases of vector and covector fields, together with the adapted components of the N-linear connections and their corresponding d-torsions and d-curvatures. But, because $L(t, x^i, y^i)$ is a real function, we deduce that the previous geometrical theory has the following impediment: -*the energy action functional depends on the reparametrizations* $t \longleftrightarrow \widetilde{t}(t)$ *of the same curve* c. Thus, in order to avoid this inconvenience, the Finsler case imposes the 1-positive homogeneity condition [3]

$$L(t, x^i, \lambda y^i) = \lambda L(t, x^i, y^i), \ \forall \ \lambda > 0.$$

Alternatively, the *relativistic non-autonomous Lagrangian geometry* from [14] uses the *relativistic energy action functional*

$$\mathbb{E}_2(c) = \int_a^b L(t, x^i, y_1^i)\sqrt{h_{11}(t)}dt,$$

where $L : J^1(\mathbb{R}, M) \to \mathbb{R}$ is a jet Lagrangian function and $h_{11}(t)$ is a Riemannian metric on the relativistic time manifold $\mathbb{R}$. This functional is now independent by the reparametrizations $t \longleftrightarrow \widetilde{t}(t)$ of the same curve c, even if L is only a function. The Euler-Lagrange equations of the *Lagrangian*

$$\mathcal{L} = L(t, x^i, y_1^i)\sqrt{h_{11}(t)}$$

produce a relativistic time dependent semispray [14]

$$\mathcal{S} = \left(H_{(1)1}^{(i)}, \ G_{(1)1}^{(i)}\right),$$

which gives the jet nonlinear connection [11]

$$\Gamma_{\mathcal{S}} = \left(M_{(1)1}^{(j)} = 2H_{(1)1}^{(j)}, \ N_{(1)k}^{(j)} = \frac{\partial G_{(1)1}^{(j)}}{\partial y_1^k}\right).$$

With these geometrical tools we can construct in the relativistic non-autonomous Lagrangian geometry the distinguished (d-) linear connections, together with their d-torsions and d-curvatures, which naturally generalize the similar geometrical objects from the classical non-autonomous Lagrangian geometry [9].

In this respect, the author of this paper believes that the jet relativistic geometrical approach proposed in the papers [11], [14], [15] has more geometrical and physical meanings than the theory proposed by Miron and Anastasiei in [9]. For such a reason, the aim of this paper is to describe the Bianchi identities that govern the jet relativistic non-autonomous Lagrangian geometry [14]. These

Bianchi identities are necessary for the construction of the *generalized Maxwell equations* that characterize the electromagnetic theory in the jet relativistic non-autonomous Lagrangian geometrical background.

In conclusion, in order to remark the main similitudes and differences between these geometrical theories, we invite the reader to compare the *classical* and *relativistic non-autonomous Lagrangian geometries* exposed in the works [9] and [14].

As a final remark, we point out that for a lot of mathematicians (such as Crampin [5], Krupková [7], de Léon [8], Sarlet [16] and others) the non-autonomous Lagrangian geometry is constructed on the first jet bundle $J^1\pi$ of a fibered manifold $\pi : M^{n+1} \longrightarrow \mathbb{R}$. In their works, if (t, x^i) are the local coordinates on the $n+1$-dimensional manifold M such that t is a global coordinate for the fibers of the submersion π and x^i are transverse coordinates of the induced foliation, then a change of coordinates on M is given by

$$\begin{cases} \widetilde{t} = \widetilde{t}(t), & \dfrac{d\widetilde{t}}{dt} \neq 0 \\ \widetilde{x}^i = \widetilde{x}^i(x^j, t), & \operatorname{rank}\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \end{cases} \tag{1.3}$$

Altough the 1-jet extension of the transformation rules (1.3) is more general than the transformation group (1.1), the author ot this paper considers that the transformation group (1.1) is nevertheless more appropriate for his final purpose, namely: – the development of a *relativistic non-autonomous Lagrangian gravitational and electromagnetic field theory* which to be characterized by some *generalized Einstein and Maxwell equations* [14]. In this direction, we need the *local Bianchi identities* that govern the jet relativistic non-autonomous Lagrangian geometry [11], [14], [15].

2 The adapted components of the jet Γ-linear connections

Let us suppose that the 1-jet space $E = J^1(\mathbb{R}, M)$ is endowed with a nonlinear connection

$$\Gamma = \left(M^{(i)}_{(1)1}, N^{(i)}_{(1)j}\right), \tag{2.1}$$

where the local coefficients $M^{(i)}_{(1)1}$ (resp., $N^{(i)}_{(1)j}$) are called the *temporal* (resp., *spatial*) *components of* Γ. Note that the transformation rules of the local components of the nonlinear connection are expressed by [11]

$$\begin{aligned} \widetilde{M}^{(k)}_{(1)1} &= M^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2 \frac{\partial \widetilde{x}^k}{\partial x^j} - \frac{dt}{d\widetilde{t}}\frac{\partial \widetilde{y}^k_1}{\partial t}, \\ \widetilde{N}^{(k)}_{(1)l} &= N^{(j)}_{(1)i}\frac{dt}{d\widetilde{t}}\frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{y}^k_1}{\partial x^i}. \end{aligned} \tag{2.2}$$

Example 2.1 *Let us consider $h = (h_{11}(t))$ (resp., $\varphi = (\varphi_{ij}(x))$) a Riemannian metric on the temporal manifold $\mathbb{R}$ (resp., the spatial manifold M) and let*

$$\varkappa_{11}^1 = \frac{h^{11}}{2}\frac{dh_{11}}{dt}, \qquad \gamma_{jk}^i = \frac{\varphi^{im}}{2}\left(\frac{\partial\varphi_{jm}}{\partial x^k} + \frac{\partial\varphi_{km}}{\partial x^j} - \frac{\partial\varphi_{jk}}{\partial x^m}\right),$$

where $h^{11} = 1/h_{11}$, be their Christoffel symbols. Then, the set of local functions

$$\mathring{\Gamma} = \left(\mathring{M}_{(1)1}^{(j)}, \mathring{N}_{(1)i}^{(j)}\right),$$

where

$$\mathring{M}_{(1)1}^{(j)} = -\varkappa_{11}^1 y_1^j, \qquad \mathring{N}_{(1)i}^{(j)} = \gamma_{im}^j y_1^m, \tag{2.3}$$

*represents a nonlinear connection on the 1-jet space $J^1(\mathbb{R}, M)$. This jet nonlinear connection is called the **canonical nonlinear connection attached to the pair of Riemannian metrics** $(h_{11}(t), \varphi_{ij}(x))$.*

In the sequel, starting with the fixed nonlinear connection Γ given by (2.1), we construct the *horizontal* vector fields

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - M_{(1)1}^{(j)}\frac{\partial}{\partial y_1^j}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^{(j)}\frac{\partial}{\partial y_1^j}, \tag{2.4}$$

and the *vertical* covector fields

$$\delta y_1^i = dy_1^i + M_{(1)1}^{(i)}dt + N_{(1)j}^{(i)}dx^j. \tag{2.5}$$

It is easy to see now that the set of vector fields

$$\left\{\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_1^i}\right\} \subset \mathcal{X}(E) \tag{2.6}$$

represents a *basis* in the set of vector fields on $J^1(\mathbb{R}, M)$, and the set of covector fields

$$\left\{dt, dx^i, \delta y_1^i\right\} \subset \mathcal{X}^*(E) \tag{2.7}$$

represents its *dual basis* in the set of 1-forms on $J^1(\mathbb{R}, M)$.

Definition 2.2 *The dual bases (2.6) and (2.7) are called the **adapted bases** attached to the nonlinear connection Γ on the 1-jet space $E = J^1(\mathbb{R}, M)$.*

Remark 2.3 *It is important to note that the local transformation laws of the elements of the adapted bases (2.6) and (2.7) are classical tensorial ones:*

$$\begin{aligned} &\frac{\delta}{\delta t} = \frac{d\widetilde{t}}{dt}\frac{\delta}{\delta\widetilde{t}}, \quad \frac{\delta}{\delta x^i} = \frac{\partial\widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta\widetilde{x}^j}, \quad \frac{\partial}{\partial y_1^i} = \frac{\partial\widetilde{x}^j}{\partial x^i}\frac{dt}{d\widetilde{t}}\frac{\partial}{\partial\widetilde{y}_1^j}, \\ &dt = \frac{dt}{d\widetilde{t}}d\widetilde{t}, \quad dx^i = \frac{\partial x^i}{\partial\widetilde{x}^j}d\widetilde{x}^j, \quad \delta y_1^i = \frac{\partial x^i}{\partial\widetilde{x}^j}\frac{d\widetilde{t}}{dt}\delta\widetilde{y}_1^j. \end{aligned} \tag{2.8}$$

For such a reason, is our choice to describe the geometrical objects of the 1-jet space $E = J^1(\mathbb{R}, M)$ in local adapted components.

It is obvious that the Lie algebra $\mathcal{X}(E)$ of the vector fields on $E = J^1(\mathbb{R}, M)$ decomposes as $\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_\mathbb{R}) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V})$, where

$$\mathcal{X}(\mathcal{H}_\mathbb{R}) = Span\left\{\frac{\delta}{\delta t}\right\}, \quad \mathcal{X}(\mathcal{H}_M) = Span\left\{\frac{\delta}{\delta x^i}\right\}, \quad \mathcal{X}(\mathcal{V}) = Span\left\{\frac{\partial}{\partial y_1^i}\right\}.$$

Let us denote as $h_\mathbb{R}$, h_M (horizontal) and v (vertical) the canonical projections produced by the above decomposition.

Definition 2.4 *A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$, which verifies the conditions $\nabla h_\mathbb{R} = 0$, $\nabla h_M = 0$ and $\nabla v = 0$, is called a* ***Γ-linear connection*** *on the 1-jet space $E = J^1(\mathbb{R}, M)$. Obviously, the local description of a Γ-linear connection ∇ on E is given by a set of* ***nine*** *local adapted components*

$$\nabla\Gamma = \left(\bar{G}^1_{11},\ G^k_{i1},\ G^{(i)(1)}_{(1)(j)1},\ \bar{L}^1_{1j},\ L^k_{ij},\ L^{(i)(1)}_{(1)(j)k},\ \bar{C}^{1(1)}_{1(k)},\ C^{j(1)}_{i(k)},\ C^{(i)(1)(1)}_{(1)(j)(k)}\right), \quad (2.9)$$

which are produced by the relations:

$$(h_\mathbb{R}) \quad \nabla_{\frac{\delta}{\delta t}}\frac{\delta}{\delta t} = \bar{G}^1_{11}\frac{\delta}{\delta t}, \quad \nabla_{\frac{\delta}{\delta t}}\frac{\delta}{\delta x^i} = G^k_{i1}\frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta t}}\frac{\partial}{\partial y_1^i} = G^{(k)(1)}_{(1)(i)1}\frac{\partial}{\partial y_1^k},$$

$$(h_M) \quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta t} = \bar{L}^1_{1j}\frac{\delta}{\delta t}, \quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = L^k_{ij}\frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y_1^i} = L^{(k)(1)}_{(1)(i)j}\frac{\partial}{\partial y_1^k},$$

$$(v) \quad \nabla_{\frac{\partial}{\partial y_1^j}}\frac{\delta}{\delta t} = \bar{C}^{1(1)}_{1(j)}\frac{\delta}{\delta t}, \quad \nabla_{\frac{\partial}{\partial y_1^j}}\frac{\delta}{\delta x^i} = C^{k(1)}_{i(j)}\frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\partial}{\partial y_1^j}}\frac{\partial}{\partial y_1^i} = C^{(k)(1)(1)}_{(1)(i)(j)}\frac{\partial}{\partial y_1^k}.$$

Example 2.5 *Let us consider the nonlinear connection $\mathring{\Gamma}$ given by (2.3), produced by the pair of Riemannian metrics $(h_{11}(t), \varphi_{ij}(x))$. Then, the set of adapted local components*

$$B\mathring{\Gamma} = \left(\bar{G}^1_{11},\ 0,\ G^{(k)(1)}_{(1)(i)1},\ 0,\ L^k_{ij},\ L^{(k)(1)}_{(1)(i)j},\ 0,\ 0,\ 0\right),$$

where

$$\bar{G}^1_{11} = \varkappa^1_{11}, \quad G^{(k)(1)}_{(1)(i)1} = -\delta^k_i\varkappa^1_{11}, \quad L^k_{ij} = \gamma^k_{ij}, \quad L^{(k)(1)}_{(1)(i)j} = \gamma^k_{ij},$$

defines a $\mathring{\Gamma}$-linear connection on the 1-jet space $E = J^1(\mathbb{R}, M)$. This connection is called the ***Berwald linear connection attached to the Riemannian metrics*** *$h_{11}(t)$* ***and*** *$\varphi_{ij}(x)$.*

Now, let ∇ be a fixed Γ-linear connection on the 1-jet space $E = J^1(\mathbb{R}, M)$, given by the adapted local components (2.9).

Definition 2.6 *A geometrical object $D = \left(D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\right)$ on the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$, whose local components transform by the rules*

$$D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots} = \widetilde{D}^{1p(m)(1)\ldots}_{1r(1)(s)\ldots}\frac{dt}{d\widetilde{t}}\frac{\partial x^i}{\partial\widetilde{x}^p}\left(\frac{\partial x^j}{\partial\widetilde{x}^m}\frac{d\widetilde{t}}{dt}\right)\frac{d\widetilde{t}}{dt}\frac{\partial\widetilde{x}^r}{\partial x^k}\left(\frac{\partial\widetilde{x}^s}{\partial x^l}\frac{dt}{d\widetilde{t}}\right)\ldots,$$

is called a ***d-tensor field****.*

Example 2.7 *If $h_{11}(t)$ is a Riemannian metric on the time manifold $\mathbb{R}$, then the geometrical object $\mathbf{J} = \left(\mathbf{J}^{(i)}_{(1)1j}\right)$, where $\mathbf{J}^{(i)}_{(1)1j} = h_{11}\delta^i_j$, represents a d-tensor field on the 1-jet space $E = J^1(\mathbb{R}, M)$. This is called the h-**normalization d-tensor field**.*

The Γ-linear connection ∇ naturally induces a linear connection on the set of the d-tensors of the 1-jet vector bundle E, in the following way: $-$ starting with $X \in \mathcal{X}(E)$ a vector field and D a d-tensor field on E, locally expressed by

$$X = X^1\frac{\delta}{\delta t} + X^r\frac{\delta}{\delta x^r} + X^{(r)}_{(1)}\frac{\partial}{\partial y^r_1},$$
$$D = D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\frac{\delta}{\delta t}\otimes\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^j_1}\otimes dt\otimes dx^k\otimes\delta y^l_1\ldots,$$

we introduce the covariant derivative

$$\nabla_X D = X^1\nabla_{\frac{\delta}{\delta t}}D + X^p\nabla_{\frac{\delta}{\delta x^p}}D + X^{(p)}_{(1)}\nabla_{\frac{\partial}{\partial y^p_1}}D = \Big\{X^1 D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots/1} + X^p\cdot$$
$$\cdot D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots|p} + X^{(p)}_{(1)}D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}|^{(1)}_{(p)}\Big\}\frac{\delta}{\delta t}\otimes\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^j_1}\otimes dt\otimes dx^k\otimes\delta y^l_1\ldots,$$

where

$$(h_{\mathbb{R}})\quad\begin{cases} D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots/1} = \dfrac{\delta D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}}{\delta t} + D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{G}^1_{11} + \\ + D^{1r(j)(1)\ldots}_{1k(1)(l)\ldots}G^i_{r1} + D^{1i(r)(1)\ldots}_{1k(1)(l)\ldots}G^{(j)(1)}_{(1)(r)1} + \ldots - \\ - D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{G}^1_{11} - D^{1i(j)(1)\ldots}_{1r(1)(l)\ldots}G^r_{k1} - D^{1i(j)(1)\ldots}_{1k(1)(r)\ldots}G^{(r)(1)}_{(1)(l)1} - \ldots, \end{cases}$$

$$(h_M)\quad\begin{cases} D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots|p} = \dfrac{\delta D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}}{\delta x^p} + D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{L}^1_{1p} + \\ + D^{1r(j)(1)\ldots}_{1k(1)(l)\ldots}L^i_{rp} + D^{1i(r)(1)\ldots}_{1k(1)(l)\ldots}L^{(j)(1)}_{(1)(r)p} + \ldots - \\ - D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{L}^1_{1p} - D^{1i(j)(1)\ldots}_{1r(1)(l)\ldots}L^r_{kp} - D^{1i(j)(1)\ldots}_{1k(1)(r)\ldots}L^{(r)(1)}_{(1)(l)p} - \ldots, \end{cases}$$

$$(v)\quad\begin{cases} D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}|^{(1)}_{(p)} = \dfrac{\partial D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}}{\partial y^p_1} + D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{C}^{1(1)}_{1(p)} + \\ + D^{1r(j)(1)\ldots}_{1k(1)(l)\ldots}C^{i(1)}_{r(p)} + D^{1i(r)(1)\ldots}_{1k(1)(l)\ldots}C^{(j)(1)(1)}_{(1)(r)(p)} + \ldots - \\ - D^{1i(j)(1)\ldots}_{1k(1)(l)\ldots}\bar{C}^{1(1)}_{1(p)} - D^{1i(j)(1)\ldots}_{1r(1)(l)\ldots}C^{r(1)}_{k(p)} - D^{1i(j)(1)\ldots}_{1k(1)(r)\ldots}C^{(r)(1)(1)}_{(1)(l)(p)} - \ldots. \end{cases}$$

Definition 2.8 *The local derivative operators "$_{/1}$", "$_{|p}$" and "$|^{(1)}_{(p)}$" are called the $\mathbb{R}$-**horizontal**, M-**horizontal** and **vertical covariant derivatives produced by the** Γ-**linear connection** $\nabla\Gamma$.*

3 h-Normal Γ-linear connections

The big number (nine) of components that characterize a general Γ-linear connection ∇ on the 1-jet space $E = J^1(\mathbb{R}, M)$ determines us to consider the following geometrical concept:

Definition 3.1 *A Γ-linear connection ∇ on E, whose local components (2.9) verify the relations*

$$\bar{G}^1_{11} = \varkappa^1_{11}, \quad \bar{L}^1_{1j} = 0, \quad \bar{C}^{1(1)}_{1(k)} = 0, \quad \nabla \mathbf{J} = 0,$$

*where $h = (h_{11}(t))$ is a Riemannian metric on $\mathbb{R}$, $\varkappa^1_{11}$ is its Christoffel symbol and $\mathbf{J}$ is the h-normalization d-tensor field, is called an **h-normal Γ-linear connection** on E.*

Remark 3.2 *The condition $\nabla \mathbf{J} = 0$ is equivalent with the local equalities*

$$\mathbf{J}^{(i)}_{(1)1j/1} = 0, \quad \mathbf{J}^{(i)}_{(1)1j|k} = 0, \quad \mathbf{J}^{(i)}_{(1)1j}|^{(1)}_{(k)} = 0,$$

where "$_{/1}$", "$_{|k}$" and "$|^{(1)}_{(k)}$" represent the $\mathbb{R}$-horizontal, M-horizontal and vertical local covariant derivatives produced by the Γ-linear connection $\nabla\Gamma$.

In this context, we can prove the following important local geometrical result:

Theorem 3.3 *The components of an h-normal Γ-linear connection ∇ verify the identities:*

$$\begin{array}{lll} \bar{G}^1_{11} = \varkappa^1_{11}, & \bar{L}^1_{1j} = 0, & \bar{C}^{1(1)}_{1(k)} = 0, \\ G^{(k)(1)}_{(1)(i)1} = G^k_{i1} - \delta^k_i \varkappa^1_{11}, & L^{(k)(1)}_{(1)(i)j} = L^k_{ij}, & C^{(k)(1)(1)}_{(1)(i)(j)} = C^{k(1)}_{i(j)}. \end{array} \tag{3.1}$$

Proof. The first three relations from (3.1) are a direct consequence of the definition of an h-normal Γ-linear connection ∇.

The condition $\nabla \mathbf{J} = 0$ implies the local relations

$$h_{11} G^{(i)(1)}_{(1)(j)1} = h_{11} G^i_{j1} + \delta^i_j \left[\varkappa_{111} - \frac{dh_{11}}{dt} \right],$$

$$h_{11} L^{(i)(1)}_{(1)(j)k} = h_{11} L^i_{jk}, \quad h_{11} C^{(i)(1)(1)}_{(1)(j)(k)} = h_{11} C^{i(1)}_{j(k)},$$

where $\varkappa_{111} = \varkappa^1_{11} h_{11}$ represent the Christoffel symbols of first kind attached to the Riemannian metric $h_{11}(t)$. Contracting the above relations with the inverse $h^{11} = 1/h_{11}$, we obtain the last three identities from (3.1). ∎

Remark 3.4 *The Theorem 3.3 implies that an h-normal Γ-linear connection ∇ is determined by **four** effective local components (instead of **two** effective local components for an N-linear connection in the Miron-Anastasiei case [9] or [10], pp. 21), namely*

$$\nabla\Gamma = \left(\varkappa^1_{11}, G^k_{i1}, L^k_{ij}, C^{k(1)}_{i(j)} \right). \tag{3.2}$$

The other five components of ∇ cancel or depend by the above four components, via the formulas (3.1).

Example 3.5 *The canonical Berwald $\mathring{\Gamma}$-linear connection associated to the pair of Riemannian metrics $(h_{11}(t), \varphi_{ij}(x))$ is an h-normal $\mathring{\Gamma}$-linear connection, defined by the local components $B\mathring{\Gamma} = \left(\varkappa^1_{11}, 0, \gamma^k_{ij}, 0\right)$.*

The study of adapted components of the torsion tensor $\mathbf{T}$ and curvature tensor $\mathbf{R}$ of an arbitrary Γ-linear connection ∇ on $E = J^1(\mathbb{R}, M)$ was completely done in the paper [15]. In that paper, we proved that the torsion tensor $\mathbf{T}$ is determined by *ten* effective local d-tensors, while the curvature tensor $\mathbf{R}$ is determined by *fifteen* effective local d-tensors. In the sequel, we study the adapted components of the torsion and curvature tensors for an h-normal Γ-linear connection ∇ given by (3.2) and (3.1).

Theorem 3.6 *The torsion tensor $\mathbf{T}$ of an h-normal Γ-linear connection ∇ on E is determined by the following **eight** adapted local d-tensors (instead of **ten** in the general case [15], pp. 12):*

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	T^r_{1j}	$R^{(r)}_{(1)1j}$
$h_M h_M$	0	T^r_{ij}	$R^{(r)}_{(1)ij}$
$vh_{\mathbb{R}}$	0	0	$P^{(r)\ (1)}_{(1)1(j)}$
vh_M	0	$P^{r(1)}_{i(j)}$	$P^{(r)\ (1)}_{(1)i(j)}$
vv	0	0	$S^{(r)(1)(1)}_{(1)(i)(j)}$

(3.3)

where

1. $T^r_{1j} = -G^r_{j1}$,

2. $R^{(r)}_{(1)1j} = \dfrac{\delta M^{(r)}_{(1)1}}{\delta x^j} - \dfrac{\delta N^{(r)}_{(1)j}}{\delta t}$,

3. $T^r_{ij} = L^r_{ij} - L^r_{ji}$,

4. $R^{(r)}_{(1)ij} = \dfrac{\delta N^{(r)}_{(1)i}}{\delta x^j} - \dfrac{\delta N^{(r)}_{(1)j}}{\delta x^i}$,

5. $P^{(r)\ (1)}_{(1)1(j)} = \dfrac{\partial M^{(r)}_{(1)1}}{\partial y^j_1} - G^r_{j1} + \delta^r_j \varkappa^1_{11}$,

6. $P^{r(1)}_{i(j)} = C^{r(1)}_{i(j)}$,

7. $P^{(r)\ (1)}_{(1)i(j)} = \dfrac{\partial N^{(r)}_{(1)i}}{\partial y^j_1} - L^r_{ji}$,

8. $S^{(r)(1)(1)}_{(1)(i)(j)} = C^{r(1)}_{i(j)} - C^{r(1)}_{j(i)}$.

Proof. Particularizing the general local expressions from [15] (which give those ten components of the torsion tensor of a Γ-linear connection ∇, in the large) for an h-normal Γ-linear connection ∇, we deduce that the adapted local components $\bar{T}^1_{1j}$ and $\bar{P}^{1(1)}_{1(j)}$ vanish, while the other eight ones from the Table 3.3 are expressed by the preceding formulas. ∎

Remark 3.7 *The torsion of an N-linear connection in the Miron-Anastasiei case is characterized only by* ***five*** *effective adapted components (please see [9] or [10], pp. 24).*

Remark 3.8 *For the Berwald $\mathring{\Gamma}$-linear connection $B\mathring{\Gamma}$ associated to the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, all adapted local torsion d-tensors vanish, except*

$$R^{(k)}_{(1)ij} = \mathfrak{R}^k_{mij} y^m_1,$$

where $\mathfrak{R}^k_{mij}(x)$ are the classical local curvature components of the Riemannian metric $\varphi_{ij}(x)$.

The expressions of the local curvature d-tensors of an arbitrary Γ-linear connection, together with the particular properties of an h-normal Γ-linear connection, imply a considerable reduction (from *fifteen* to *five*) of the effective local curvature d-tensors that characterize an h-normal Γ-linear connection. In other words, we have

Theorem 3.9 *The curvature tensor $\mathbf{R}$ of an h-normal Γ-linear connection ∇ on E is characterized by* ***five*** *effective local curvature d-tensors (instead of* ***fifteen*** *in the general case [15], pp. 14):*

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	R^l_{i1k}	$R^{(l)(1)}_{(1)(i)1k} = R^l_{i1k}$
$h_M h_M$	0	R^l_{ijk}	$R^{(l)(1)}_{(1)(i)jk} = R^l_{ijk}$
$vh_{\mathbb{R}}$	0	$P^{l\ \ (1)}_{i1(k)}$	$P^{(l)(1)\ (1)}_{(1)(i)1(k)} = P^{l\ \ (1)}_{i1(k)}$
vh_M	0	$P^{l\ (1)}_{ij(k)}$	$P^{(l)(1)\ (1)}_{(1)(i)j(k)} = P^{l\ (1)}_{ij(k)}$
vv	0	$S^{l(1)(1)}_{i(j)(k)}$	$S^{(l)(1)(1)(1)}_{(1)(i)(j)(k)} = S^{l(1)(1)}_{i(j)(k)}$

(3.4)

where

1. $R^l_{i1k} = \dfrac{\delta G^l_{i1}}{\delta x^k} - \dfrac{\delta L^l_{ik}}{\delta t} + G^r_{i1}L^l_{rk} - L^r_{ik}G^l_{r1} + C^{l(1)}_{i(r)}R^{(r)}_{(1)1k},$

2. $R^l_{ijk} = \dfrac{\delta L^l_{ij}}{\delta x^k} - \dfrac{\delta L^l_{ik}}{\delta x^j} + L^r_{ij}L^l_{rk} - L^r_{ik}L^l_{rj} + C^{l(1)}_{i(r)}R^{(r)}_{(1)jk},$

3. $P^{l\ \ (1)}_{i1(k)} = \dfrac{\partial G^l_{i1}}{\partial y^k_1} - C^{l(1)}_{i(k)/1} + C^{l(1)}_{i(r)}P^{(r)\ \ (1)}_{(1)1(k)},$

4. $P^{l\ (1)}_{ij(k)} = \dfrac{\partial L^l_{ij}}{\partial y^k_1} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(r)} P^{(r)\ (1)}_{(1)j(k)},$

5. $S^{l(1)(1)}_{i(j)(k)} = \dfrac{\partial C^{l(1)}_{i(j)}}{\partial y^k_1} - \dfrac{\partial C^{l(1)}_{i(k)}}{\partial y^j_1} + C^{r(1)}_{i(j)} C^{l(1)}_{r(k)} - C^{r(1)}_{i(k)} C^{l(1)}_{r(j)}.$

Proof. The general formulas that express those fifteen local curvature d-tensors of an arbitrary Γ-linear connection [15], applied to the particular case of an h-normal Γ-linear connection ∇ on E, imply the preceding formulas and the relations from the Table 3.4. ■

Remark 3.10 *The curvature of an N-linear connection in the Miron-Anastasiei case is characterized only by* ***three*** *effective adapted components (please see [9] or [10], pp. 25).*

Remark 3.11 *For the Berwald $\mathring{\Gamma}$-linear connection $B\mathring{\Gamma}$ associated to the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, all local curvature d-tensors vanish, except*

$$R^{(l)(1)}_{(1)(i)jk} = R^l_{ijk} = \mathfrak{R}^l_{ijk},$$

where $\mathfrak{R}^l_{ijk}(x)$ are the local curvature tensors of the Riemannian metric $\varphi_{ij}(x)$.

4 d-Connections of Cartan type. Local Ricci and Bianchi identities

Because of the reduced number and the simplified form of the local torsion and curvature d-tensors of an h-normal Γ-linear connection ∇ on the 1-jet space E, the number of attached *local Ricci and Bianchi identities* considerably simplifies. A substantial reduction of these identities obtains considering the more particular case of an h-normal Γ-linear connection of *Cartan type*.

Definition 4.1 *An h-normal Γ-linear connection on $E = J^1(\mathbb{R}, M)$, whose local components*

$$\nabla\Gamma = \left(\varkappa^1_{11}, G^k_{i1}, L^k_{ij}, C^{k(1)}_{i(j)}\right)$$

verify the supplementary conditions $L^k_{ij} = L^k_{ji}$ and $C^{k(1)}_{i(j)} = C^{k(1)}_{j(i)}$, is called an ***$h$-normal Γ-linear connection of Cartan type****.*

Remark 4.2 *In the particular case of an h-normal Γ-linear connection of Cartan type, the conditions $L^k_{ij} = L^k_{ji}$ and $C^{k(1)}_{i(j)} = C^{k(1)}_{j(i)}$ imply the torsion equalities*

$$T^k_{ij} = 0, \quad S^{(k)(1)(1)}_{(1)(i)(j)} = 0.$$

Rewriting the local Ricci identities of a Γ-linear connection ∇ (described in the large in [15], pp. 15), for the particular case of an h-normal Γ-linear connection of Cartan type, we find a simplified form of these identities. Consequently, we obtain

Theorem 4.3 *The following **local Ricci identities** for an h-normal Γ-linear connection of Cartan type are true:*

$$(h_{\mathbb{R}})\quad \begin{cases} X^1_{/1|k} - X^1_{|k/1} = -X^1_{|r} T^r_{1k} - X^1|^{(1)}_{(r)} R^{(r)}_{(1)1k} \\ X^1_{|j|k} - X^1_{|k|j} = -X^1|^{(1)}_{(r)} R^{(r)}_{(1)jk} \\ X^1_{/1}|^{(1)}_{(k)} - X^1|^{(1)}_{(k)/1} = -X^1|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)1(k)} \\ X^1_{|j}|^{(1)}_{(k)} - X^1|^{(1)}_{(k)|j} = -X^1_{|r} C^{r(1)}_{j(k)} - X^1|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)j(k)} \\ X^1|^{(1)}_{(j)}|^{(1)}_{(k)} - X^1|^{(1)}_{(k)}|^{(1)}_{(j)} = 0, \end{cases}$$

$$(h_M)\quad \begin{cases} X^i_{/1|k} - X^i_{|k/1} = X^r R^i_{r1k} - X^i_{|r} T^r_{1k} - X^i|^{(1)}_{(r)} R^{(r)}_{(1)1k} \\ X^i_{|j|k} - X^i_{|k|j} = X^r R^i_{rjk} - X^i|^{(1)}_{(r)} R^{(r)}_{(1)jk} \\ X^i_{/1}|^{(1)}_{(k)} - X^i|^{(1)}_{(k)/1} = X^r P^{i\ (1)}_{r1(k)} - X^i|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)1(k)} \\ X^i_{|j}|^{(1)}_{(k)} - X^i|^{(1)}_{(k)|j} = X^r P^{i\ (1)}_{rj(k)} - X^i_{|r} C^{r(1)}_{j(k)} - X^i|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)j(k)} \\ X^i|^{(1)}_{(j)}|^{(1)}_{(k)} - X^i|^{(1)}_{(k)}|^{(1)}_{(j)} = X^r S^{i(1)(1)}_{r(j)(k)}, \end{cases}$$

$$(v)\quad \begin{cases} X^{(i)}_{(1)/1|k} - X^{(i)}_{(1)|k/1} = X^{(r)}_{(1)} R^i_{r1k} - X^{(i)}_{(1)|r} T^r_{1k} - X^{(i)}_{(1)}|^{(1)}_{(r)} R^{(r)}_{(1)1k} \\ X^{(i)}_{(1)|j|k} - X^{(i)}_{(1)|k|j} = X^{(r)}_{(1)} R^i_{rjk} - X^{(i)}_{(1)}|^{(1)}_{(r)} R^{(r)}_{(1)jk} \\ X^{(i)}_{(1)/1}|^{(1)}_{(k)} - X^{(i)}_{(1)}|^{(1)}_{(k)/1} = X^{(r)}_{(1)} P^{i\ (1)}_{r1(k)} - X^{(i)}_{(1)}|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)1(k)} \\ X^{(i)}_{(1)|j}|^{(1)}_{(k)} - X^{(i)}_{(1)}|^{(1)}_{(k)|j} = X^{(r)}_{(1)} P^{i\ (1)}_{rj(k)} - X^{(i)}_{(1)|r} C^{r(1)}_{j(k)} - X^{(i)}_{(1)}|^{(1)}_{(r)} P^{(r)\ (1)}_{(1)j(k)} \\ X^{(i)}_{(1)}|^{(1)}_{(j)}|^{(1)}_{(k)} - X^{(i)}_{(1)}|^{(1)}_{(k)}|^{(1)}_{(j)} = X^{(r)}_{(1)} S^{i(1)(1)}_{r(j)(k)}, \end{cases}$$

where

$$X = X^1 \frac{\delta}{\delta t} + X^i \frac{\delta}{\delta x^i} + X^{(i)}_{(1)} \frac{\partial}{\partial y^i_1}$$

is an arbitrary distinguished vector field on the 1-jet space $E = J^1(\mathbb{R}, M)$.

In what follows, let us consider the *canonical jet Liouville d-vector field*

$$\mathbf{C} = \mathbf{C}^{(i)}_{(1)} \frac{\partial}{\partial y^i_1} = y^i_1 \frac{\partial}{\partial y^i_1},$$

and let us construct the *nonmetrical deflection d-tensors*

$$\bar{D}^{(i)}_{(1)1} = \mathbf{C}^{(i)}_{(1)/1}, \quad D^{(i)}_{(1)j} = \mathbf{C}^{(i)}_{(1)|j}, \quad d^{(i)(1)}_{(1)(j)} = \mathbf{C}^{(i)}_{(1)}|^{(1)}_{(j)}.$$

Then, a direct calculation leads to

Proposition 4.4 *The nonmetrical deflection d-tensors attached to the h-normal Γ-linear connection ∇ given by (3.2) have the expressions*

$$\begin{gathered}\bar{D}^{(i)}_{(1)1} = -M^{(i)}_{(1)1} + G^i_{r1}y^r_1 - \varkappa^1_{11}y^i_1, \qquad D^{(i)}_{(1)j} = -N^{(i)}_{(1)j} + L^i_{rj}y^r_1,\\ d^{(i)(1)}_{(1)(j)} = \delta^i_j + C^{i(1)}_{r(j)}y^r_1.\end{gathered} \tag{4.1}$$

Applying now the preceding $(v)-$ set of local Ricci identities (associated to an h-normal Γ-linear connection of Cartan type) to the components of the canonical jet Liouville d-vector field, we find

Theorem 4.5 *The following **five** identities of the nonmetrical deflection d-tensors associated to an h-normal Γ-linear connection of Cartan type (instead of **three** in the Miron-Anastasiei's case [9] or [10], pp. 80) are true:*

$$\begin{cases}\bar{D}^{(i)}_{(1)1|k} - D^{(i)}_{(1)k/1} = y^r_1 R^i_{r1k} - D^{(i)}_{(1)r}T^r_{1k} - d^{(i)(1)}_{(1)(r)}R^{(r)}_{(1)1k}\\ D^{(i)}_{(1)j|k} - D^{(i)}_{(1)k|j} = y^r_1 R^i_{rjk} - d^{(i)(1)}_{(1)(r)}R^{(r)}_{(1)jk}\\ \bar{D}^{(i)}_{(1)1}|^{(1)}_{(k)} - d^{(i)(1)}_{(1)(k)/1} = y^r_1 P^{i\ \ (1)}_{r1(k)} - d^{(i)(1)}_{(1)(r)}P^{(r)\ \ (1)}_{(1)1(k)}\\ D^{(i)}_{(1)j}|^{(1)}_{(k)} - d^{(i)(1)}_{(1)(k)|j} = y^r_1 P^{i\ \ (1)}_{rj(k)} - D^{(i)}_{(1)r}C^{r(1)}_{j(k)} - d^{(i)(1)}_{(1)(r)}P^{(r)\ (1)}_{(1)j(k)}\\ d^{(i)(1)}_{(1)(j)}|^{(1)}_{(k)} - d^{(i)(1)}_{(1)(k)}|^{(1)}_{(j)} = y^r_1 S^{i(1)(1)}_{r(j)(k)}.\end{cases} \tag{4.2}$$

Remark 4.6 *The identities (4.2) are used in the description of **generalized Maxwell equations** that govern the **electromagnetic 2-form** [14] produced by a relativistic time dependent Lagrangian on the 1-jet space $E = J^1(\mathbb{R}, M)$.*

The using of h-normal Γ-linear connections of Cartan type in the study of differential geometry of the 1-jet vector bundle $E = J^1(\mathbb{R}, M)$ is also convenient because the number and the form of the local Bianchi identities associated to such connections are considerably simplified. In fact, we have:

Theorem 4.7 *The following **nineteen** effective **local Bianchi identities** for the h-normal Γ-linear connections of Cartan type ∇ given by (3.2) are true:*

1. $\mathcal{A}_{\{j,k\}}\left\{R^l_{j1k} + T^l_{1j|k} + R^{(r)}_{(1)1j}C^{l(1)}_{k(r)}\right\} = 0,$

2*. $\sum_{\{i,j,k\}}\left\{R^l_{ijk} - R^{(r)}_{(1)ij}C^{l(1)}_{k(r)}\right\} = 0,$

3. $\mathcal{A}_{\{j,k\}}\left\{R^{(l)}_{(1)1j|k} + T^r_{1j}R^{(l)}_{(1)kr} + R^{(r)}_{(1)1j}P^{(l)\ \ (1)}_{(1)k(r)}\right\} =$

$\qquad = -R^{(l)}_{(1)jk/1} - R^{(r)}_{(1)jk}P^{(l)\ \ (1)}_{(1)1(r)},$

4*. $\sum_{\{i,j,k\}}\left\{R^{(l)}_{(1)ij|k} + R^{(r)}_{(1)ij}P^{(l)\ \ (1)}_{(1)k(r)}\right\} = 0,$

5. $T^l_{1k}|^{(1)}_{(p)} - C^{l(1)}_{r(p)}T^r_{1k} + P^{l\ \ (1)}_{k1(p)} + C^{l(1)}_{k(p)/1} + C^{r(1)}_{k(p)}T^l_{1r} - C^{l(1)}_{k(r)}P^{(r)\ \ (1)}_{(1)1(p)} = 0,$

6*. $\mathcal{A}_{\{j,k\}}\left\{C^{l(1)}_{j(p)|k}+C^{l(1)}_{k(r)}P^{(r)\ \ (1)}_{(1)j(p)}+P^{l\ \ (1)}_{jk(p)}\right\}=0,$

7. $P^{(l)\ \ (1)}_{(1)1(p)|k}-P^{(l)\ \ (1)}_{(1)k(p)/1}+P^{(l)\ \ (1)}_{(1)k(r)}P^{(r)\ \ (1)}_{(1)1(p)}-P^{(l)\ \ (1)}_{(1)1(r)}P^{(r)\ \ (1)}_{(1)k(p)}=$

$$=R^{(l)}_{(1)1k}|^{(1)}_{(p)}-R^{l}_{p1k}+R^{(l)}_{(1)1r}C^{r(1)}_{k(p)}-T^{r}_{1k}P^{(l)\ \ (1)}_{(1)r(p)},$$

8*. $\mathcal{A}_{\{j,k\}}\left\{R^{(l)}_{(1)jr}C^{r(1)}_{k(p)}+P^{(l)\ \ (1)}_{(1)j(r)}P^{(r)\ \ (1)}_{(1)k(p)}+P^{(l)\ \ (1)}_{(1)k(p)|j}\right\}=R^{l}_{pjk}-R^{(l)}_{(1)jk}|^{(1)}_{(p)},$

9*. $\mathcal{A}_{\{j,k\}}\left\{C^{l(1)}_{i(j)}|^{(1)}_{(k)}+C^{r(1)}_{i(k)}C^{l(1)}_{r(j)}\right\}=$

$$=\mathcal{A}_{\{j,k\}}\left\{\frac{C^{l(1)}_{i(j)}}{\partial y^k_1}+C^{r(1)}_{i(j)}C^{l(1)}_{r(k)}\right\}=S^{l(1)(1)}_{i(j)(k)},$$

10. $\mathcal{A}_{\{j,k\}}\left\{P^{(l)\ \ (1)}_{(1)1(j)}|^{(1)}_{(k)}+P^{l\ \ (1)}_{j1(k)}\right\}=0,$

11*. $\mathcal{A}_{\{j,k\}}\left\{P^{l\ \ (1)}_{ji(k)}+P^{(l)\ \ (1)}_{(1)r(j)}C^{r(1)}_{i(k)}-P^{(l)\ (1)}_{(1)i(k)}|^{(1)}_{(j)}\right\}=0,$

12*. $\sum_{\{i,j,k\}}S^{l(1)(1)}_{i(j)(k)}=0,$

13. $\mathcal{A}_{\{j,k\}}\left\{R^{l}_{p1j|k}+T^{r}_{1j}R^{l}_{pkr}+R^{(r)}_{(1)1j}P^{l\ \ (1)}_{pk(r)}\right\}=-R^{l}_{pjk/1}-R^{(r)}_{(1)jk}P^{l\ \ (1)}_{p1(r)},$

14*. $\sum_{\{i,j,k\}}\left\{R^{l}_{pij|k}+R^{(r)}_{(1)ij}P^{l\ \ (1)}_{pk(r)}\right\}=0,$

15. $P^{l\ \ (1)}_{i1(p)|k}-P^{l\ \ (1)}_{ik(p)/1}+P^{(r)\ \ (1)}_{(1)1(p)}P^{l\ \ (1)}_{ik(r)}-P^{(r)\ \ (1)}_{(1)k(p)}P^{l\ \ (1)}_{i1(r)}=$

$$=R^{l}_{i1k}|^{(1)}_{(p)}+R^{(r)}_{(1)1k}S^{l(1)(1)}_{i(p)(r)}+C^{r(1)}_{k(p)}R^{l}_{i1r}-T^{r}_{1k}P^{l\ \ (1)}_{ir(p)},$$

16*. $\mathcal{A}_{\{j,k\}}\left\{R^{l}_{ijr}C^{r(1)}_{k(p)}+P^{l\ \ (1)}_{ij(r)}P^{(r)\ \ (1)}_{(1)k(p)}+P^{l\ \ (1)}_{ik(p)|j}\right\}=$

$$=-S^{l(1)(1)}_{i(p)(r)}R^{(r)}_{(1)jk}-R^{l}_{ijk}|^{(1)}_{(p)},$$

17. $\mathcal{A}_{\{j,k\}}\left\{P^{l\ \ (1)}_{p1(j)}|^{(1)}_{(k)}+P^{(r)\ \ (1)}_{(1)1(j)}S^{l(1)(1)}_{p(k)(r)}\right\}=-S^{l(1)(1)}_{p(j)(k)/1},$

18*. $\mathcal{A}_{\{j,k\}}\left\{P^{l\ \ (1)}_{pr(j)}C^{r(1)}_{i(k)}-S^{l(1)(1)}_{p(j)(r)}P^{(r)\ \ (1)}_{(1)i(k)}-P^{l\ \ (1)}_{pi(k)}|^{(1)}_{(j)}\right\}=-S^{l(1)(1)}_{p(j)(k)|i},$

19*. $\sum_{\{i,j,k\}}S^{l(1)(1)}_{p(i)(j)}|^{(1)}_{(k)}=0,$ *where* $\sum_{\{i,j,k\}}$ *represents a cyclic sum and* $\mathcal{A}_{\{i,j\}}$ *represents an alternate sum.*

Proof. Let $(X_A)=\left(\delta/\delta t,\delta/\delta x^i,\partial/\partial y^i_1\right)$ be the adapted basis of vector fields produced by the nonlinear connection (2.1). Let ∇ be the h-normal Γ-linear connection of Cartan type given by (3.2). For the linear connection ∇ the following general local Bianchi identities are true [9], [12]:

$$\sum_{\{A,B,C\}}\left\{\mathbf{R}^F_{ABC}-\mathbf{T}^F_{AB:C}-\mathbf{T}^G_{AB}\mathbf{T}^F_{CG}\right\}=0,$$

$$\sum_{\{A,B,C\}}\left\{\mathbf{R}^F_{DAB:C}+\mathbf{T}^G_{AB}\mathbf{R}^F_{DCG}\right\}=0,$$

where $\mathbf{R}(X_A,X_B)X_C=\mathbf{R}^D_{CBA}X_D$, $\mathbf{T}(X_A,X_B)=\mathbf{T}^D_{BA}X_D$ and "$_{:A}$" represents one from the local covariant derivatives "$_{/1}$", "$_{|i}$" or "$|^{(1)}_{(i)}$". Obviously, the components $\mathbf{T}^C_{AB}$ and $\mathbf{R}^D_{ABC}$ are the adapted components of the torsion and curvature tensors associated to the linear connection $\nabla\Gamma$. These components are expressed in the Tables 3.3 and 3.4. Then, replacing $A,B,C,\ldots$ with indices of type

$$\left\{1,i,\begin{matrix}(1)\\(i)\end{matrix}\right\},$$

by laborious local computations, we obtain the required Bianchi identities. ■

Remark 4.8 *The above **eleven "star"-Bianchi identities** are exactly those eleven Bianchi identities that characterize the canonical metrical Cartan connection of a Finsler space (please see [10], pp.48).*

Remark 4.9 *The importance of preceding Bianchi identities for an h-normal Γ-linear connection of Cartan type ∇ on the 1-jet space $J^1(\mathbb{R},M)$ comes from their using in the local description of the **generalized Maxwell equations** (please see [14], pp. 161) that characterize an electromagnetic field in the background of relativistic non-autonomous Lagrange geometry.*

References

[1] M. Anastasiei, H. Kawaguchi, *A geometrical theory of time dependent Lagrangians, I. Nonlinear connections, II. M-Connections*, Tensor N. S. **48** (1989), 273-282, 283-293.

[2] G. S. Asanov, *Jet extension of Finslerian gauge approach*, Fortschritte der Physik **38**, No. **8** (1990), 571-610.

[3] D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.

[4] I. Bucătaru, R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Romanian Academy Eds., Bucharest, 2007.

[5] M. Crampin, F.A.E. Pirani, *Applicable Differential Geometry*, Cambridge University Press, 1986.

[6] C. Frigioiu, *Noether's laws of conservation in time dependent Lagrange spaces*, Lagrange and Hamilton Geometries and Their Applications, Radu Miron (Ed.), Fair Partners Publishers, Bucharest (2004), 125-131.

[7] O. Krupková, *The Geometry of Ordinary Variational Equations*, Springer-Verlag, 1997.

[8] M. de Léon, P. Rodriguez, *Generalized Classical Mechanics and Field Theory*, North-Holland, Amsterdam, 1985.

[9] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[10] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, 2001.

[11] M. Neagu, *Jet geometrical objects depending on a relativistic time*, http://arXiv.org/math.DG/0801.2369v2; Analele Ştiinţifice ale Universităţii "Al.I.Cuza" din Iaşi. Secţiunea Matematică (2009), 1-19, in press.

[12] M. Neagu, *Ricci and Bianchi identities for h-normal Γ-linear connections on $J^1(T, M)$*, Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences, Vol. (2003), No. **34**, 2177-2192.

[13] M. Neagu, *Riemann-Lagrange Geometry on 1-Jet Spaces*, Matrix Rom, Bucharest, 2005.

[14] M. Neagu, *The geometry of relativistic rheonomic Lagrange spaces*, Proceedings of Workshop on Diff. Geom., Global Analysis and Lie Algebras, Grigorios Tsagas (Ed.), Geometry Balkan Press, Bucharest (2001), No. **5**, 142-168.

[15] M. Neagu, E. Stoica, *Deflection d-tensor identities in the relativistic time dependent Lagrange geometry*, http://arXiv.org/math.DG/0908.3592, (2009).

[16] W. Sarlet, *Geometrical structures related to second-order equations*, Proc. Conf. "*Differential Geometry and Its Applications*", Brno, Czechoslovakia, 1986; D. Krupka and A. Švec, eds.(D. Reidel, Dordrecht, 1986), 279-288.

[17] D. J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, 1989.

[18] A. Vondra, *Symmetries of connections on fibered manifolds*, Archivum Mathematicum, Tomus **30** (1994), 3-72.

Author's address:

Mircea Neagu
University Transilvania of Braşov, Faculty of Mathematics and Informatics
Department of Algebra, Geometry and Differential Equations
B-dul Iuliu Maniu, Nr. 50, BV 500091, Braşov, Romania.
E-mail: mircea.neagu@unitbv.ro
Website: http://www.2collab.com/user:mirceaneagu